Relative mapping class group of $S^p \times D^q$

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Abstract

Algebraic structure of the group of pseudo-isotopy classes of diffeomorphisms of the trivial disk bundle over the standard sphere which restrict to the identity map on the boundary is determined.

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1 Introduction

Let M be an oriented smooth manifold with non-empty boundary ∂M. We consider the group of pseudo-isotopy classes of diffeomorphisms of M which restrict to the identity map on the boundary. Recall that two diffeomorphisms $f_0, f_1 \in \mathrm{Diff}(M)$ which keep ∂M pointwise fixed are called *pseudo-isotopic (rel boundary)* if there exists a diffeomorphism $\Phi : M \times I \longrightarrow M \times I$ such that $\Phi|_{M\times\{0\}} = f_0$, $\Phi|_{M\times\{1\}} = f_1$ and $\Phi|_{\partial M \times I} = Id$. We write $f_0 \sim f_1$ to indicate that f_0 is pseudo-isotopic to f_1. Such diffeomorphisms are of course orientation preserving and if $\dim(M) = 2$, the group is known as the classical mapping class group of a surface. We will also call the group of pseudo-isotopy classes of diffeomorphisms of M which are fixed on the boundary as *the (relative) mapping class group* and denote it by $\tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial)$.

Not much is known about these groups in higher dimensions and the goal of this work was to determine such a group for the trivial disk bundles over the standard spheres (see Theorem 1 at the end). For the handlebodies in general, the pseudo-isotopy classes of diffeomorphisms with no constraint on the boundary had been studied by Wall in [14]. The corresponding group is denoted by $\tilde{\pi}_0\mathrm{Diff}(M)$. Our approach here will be based on the results of Levine [7] and Sato [10] who determined the mapping class group of $S^p \times S^q$ up to extension (cf. also work of Turner, [12]).

Integer coefficients are understood for all homology groups, unless otherwise stated, and symbols $\simeq$ and $\cong$ are used to denote diffeomorphism and isomorphism respectively. We will follow notations of [10] and denote by $\tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q)$ the subgroup of $\tilde{\pi}_0\mathrm{Diff}(S^p \times S^q)$ which consists of classes with representatives that induce trivial action on the homology. The h-cobordism classes of all homotopy m-spheres form an abelian group under the connected sum operation and we denote such a group by Θ_m (see [5] for details).

2 $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial)$

Given a manifold M with non-empty boundary, one can consider the *double* $\mathcal{D}M$ of M defined as $\mathcal{D}M := \partial(M \times I)$. $\mathcal{D}M$ is a closed manifold with the canonically defined smooth structure (cf. [8]). Since $\partial(M \times I) \simeq M \times \{0\} \cup (\partial M) \times I \cup M \times \{1\}$ and $(\partial M) \times I \cup M \times \{1\} \simeq M$ (which we will denote by M_+), one can also think of the double as of the union of two copies of M glued together along the boundary:

$$\mathcal{D}M \simeq M \cup M_+$$

For example, if $M \simeq S^p \times D^q$ then $\mathcal{D}M \simeq S^p \times S^q$. Take $\varphi \in \mathrm{Diff}(M, \mathrm{rel}\ \partial)$, then one can use the identity map to extend φ to a diffeomorphism $\tilde{\varphi}$ of $\mathcal{D}M$. To be more precise, we define

$$\tilde{\varphi}(x) := \begin{cases} \varphi(x) & \text{if} \quad x \in M \\ x & \text{if} \quad x \in M_+ \end{cases}$$

This construction gives a map $\mathrm{Diff}(M, \mathrm{rel}\ \partial) \longrightarrow \mathrm{Diff}(\mathcal{D}M)$, which induces a homomorphism $\omega : \tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial) \longrightarrow \tilde{\pi}_0\mathrm{Diff}(\mathcal{D}M)$ defined by $\omega([\varphi]) := [\tilde{\varphi}]$. The following proposition generalizes Theorem 2 of [4].

Proposition 1.
Homomorphism $\omega : \tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial) \to \tilde{\pi}_0\mathrm{Diff}(\mathcal{D}M)$ *is injective for all* M.

Proof. If $\varphi \in \tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial)$ and $\tilde{\varphi} \sim Id$, then there exists an extension $\Phi \in \mathrm{Diff}(M \times I)$ of $\tilde{\varphi} \in \mathrm{Diff}(\partial(M \times I))$. Since Φ is equal to φ on $M \times \{0\}$ and the identity map on $\partial(M) \times I \cup M \times \{1\}$, this Φ is a relative pseudo-isotopy that connects φ with Id. $\square$

It is easy to see that $\tilde{\pi}_0\mathrm{Diff}(S^p \times S^q)/\tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q) \cong \mathbb{Z}_2$ when $p \neq q$ (cf. [10], Theorem I or [7], §1.2). Since the extension diffeomorphism $\tilde{\varphi}$ fixes $S^p \times D^q_+ = M_+ \subset \mathcal{D}M$, it follows from Proposition 2.1 of [10] and proposition above that $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial)$ must be a subgroup of $\tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q)$. Suppose that $1 \leq p < q$ and $3 \leq q$, or $4 \leq p = q$, then there is a homomorphism

$$B : \tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q) \longrightarrow \pi_p(SO(q+1)).$$

defined by Sato as follows (cf. [10], §3): Take a representative f of a class $[f] \in \tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q)$ and pick a point $z \in S^q$. Then $S^p = S^p \times z \subset \mathcal{D}M$ will represent a generator of $H_p(\mathcal{D}M) \cong \mathbb{Z}$. Since f acts trivially on $H_p(\mathcal{D}M)$, it follows from the Hurewicz theorem and the result of Haefliger [2] that there exists a diffeomorphism $f' \sim f$ which is the identity on $S^p \times z$. Furthermore, if we take the tubular neighborhood $S^p \times D^q$ of the sphere $S^p \times z$, then by the tubular neighborhood theorem we can assume that f' is isotopic to f'' such that $f''(x, y) = (x, b(f) \cdot y)$, where $(x, y) \in S^p \times D^q$ and $b(f)$ is a smooth map $S^p \longrightarrow SO(q)$. Denote the inclusion map $SO(q) \hookrightarrow SO(q+1)$ by S and the homotopy class of $b(f)$ by $[b(f)] \in \pi_p(SO(q))$, then B is defined by the formula:

$$B([f]) := S_*([b(f)]).$$

For each element $[f] \in \tilde{\pi}_0\mathrm{Diff}(D^m, \mathrm{rel}\ \partial) \cong \tilde{\pi}_0\mathrm{Diff}(S^m) \cong \Theta_{m+1}$ one defines a diffeomorphism $\iota_r(f) \in \mathrm{Diff}(S^p \times D^{m-p}, \mathrm{rel}\ \partial)$ as the identity map outside an embedded disk $\mathbb{D}^m \hookrightarrow \mathrm{Int}(S^p \times D^{m-p})$ and $f|_{D^m}$ on this $\mathbb{D}^m$ (see §4 of [10] for the details). It is easily deduced from §4 of [10] that this construction induces a monomorphism $\iota_r : \Theta_{p+q+1} \hookrightarrow \tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial)$.

Furthermore, let us denote by FC^{p+1}_q, the group of the pseudo-isotopy classes of orientation preserving embeddings of $S^q \times D^{p+1}$ in S^{q+p+1}. This group was introduced by Haefliger and the reader will find all the details in

§5 of [3]. Here we only mention that $\mathrm{FC}_q^{p+1} \cong \pi_q(SO(p+1))$, when $q < 2p$ (see [3], Corollaries 5.9 & 6.6).

Lemma 1.

$$\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \cong \begin{cases} \Theta_{p+q+1} \oplus \mathrm{FC}_q^{p+1} & \text{if} \quad 1 < p < q \\ \Theta_{p+q+1} \oplus \pi_q(SO(p+1)) & \text{if} \quad 1 < q < p \end{cases}$$

Proof. Assume first that $p < q$, then we have the exact sequence (see [10], Theorem II or [7], Theorem 2.4):

$$0 \longrightarrow \mathrm{FC}_q^{p+1} \oplus \Theta_{p+q+1} \longrightarrow \tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q) \xrightarrow{B} \pi_p(SO(q+1)) \longrightarrow 0.$$

Since the diffeomorphism $\tilde{\varphi}$ fixes $S^p \times D^q_+$, it is clear from definition of B that $B([\tilde{\varphi}]) = \{0\}$, i.e. $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \subset \mathrm{Ker}(B)$. Sato had shown (see [10], Lemma 3.3) that for any element $[u] \in \mathrm{Ker}(B)$ one can find a representative $u \in \mathrm{Diff}(S^p \times S^q)$ such that $u|_{S^p \times D^q_+} = Id$, and therefore $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \cong \mathrm{Ker}(B)$.

Let now p be larger than q. In this case $\mathrm{Im}(B) = \pi_q(SO(p+1))$ and for every class $[z] \in \pi_q(SO(p+1))$ we can choose a smooth representative

$$r : (D^q, S^{q-1}) \longrightarrow (SO(p+1), Id)$$

and define a relative diffeomorphism ϑ of $S^p \times D^q$ by the formula $\vartheta(x, y) := (r(y) \cdot x, y)$, where $(x, y) \in S^p \times D^q$. Let us also use ω to denote the inclusion $\tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial) \hookrightarrow \tilde{\pi}_0 S\mathrm{Diff}(\mathcal{D}M)$. Proof of Proposition 3.2 of [10] shows that the composition $B \circ \omega$ is a surjection. If we take an element $[\varphi_0] \in \tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial)$ such that $B([\tilde{\varphi}_0]) = \{1\}$, then propositions 5.2 and 5.3 of [10] imply that $\tilde{\varphi}_0$ (modulo some element of Θ_{p+q+1} if needed) can be extended to a diffeomorphism of $S^p \times D^{q+1}$, i.e. $[\tilde{\varphi}_0] \in \Theta_{p+q+1}$ and hence the exact sequence above implies that $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \cong \pi_q(SO(p+1)) \oplus \Theta_{p+q+1}$. □

Lemma 2.

$$\tilde{\pi}_0\mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \cong \begin{cases} \{1\} & \text{if} \quad p = 1, q = 2 \\ \Theta_{q+1} \oplus \Theta_{q+2} & \text{if} \quad p = 1, q \geq 3 \\ \Theta_{p+2} \oplus \mathbb{Z}_2 & \text{if} \quad q = 1, p \geq 2 \end{cases}$$

Proof. $\tilde{\pi}_0\mathrm{Diff}(S^1 \times D^2, \mathrm{rel}\ \partial) \hookrightarrow \tilde{\pi}_0\mathrm{Diff}(S^1 \times S^2)$ by our Proposition 1. Since $\tilde{\varphi} \in \mathrm{Diff}(S^1 \times S^2)$ acts trivially on the homology, it follows from Theorem 5.1 of [1] that $\tilde{\varphi}$ is either pseudo-isotopic to the identity or the diffeomorphism T of $S^1 \times S^2$ defined by $T(t, x) := (t, f(t) \circ x)$ where $f : S^1 \longrightarrow SO(3)$ is a smooth generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$. If we had $\tilde{\varphi} \sim T$, then $\tilde{\varphi}$ would extend to a diffeomorphism of $S^1 \times D^3$, and therefore would be pseudo-isotopic to the identity map. Thus $\tilde{\pi}_0\mathrm{Diff}(S^1 \times D^2, \mathrm{rel}\ \partial) \cong \{1\}$.

Consider now $\tilde{\pi}_0\mathrm{Diff}(S^1 \times D^q, \mathrm{rel}\ \partial)$ with $q \geq 3$. Here we also can assume that $\tilde{\pi}_0\mathrm{Diff}(S^1 \times D^q, \mathrm{rel}\ \partial) \subset \mathrm{Ker}(B)$ and according to Proposition 6.3 of [10], the latter group is isomorphic to $\Theta_{q+1} \oplus \Theta_{q+2}$. For an element $[g] \in \tilde{\pi}_0\mathrm{Diff}(D^q, \mathrm{rel}\ \partial) \cong \tilde{\pi}_0\mathrm{Diff}(S^q) \cong \Theta_{q+1}$, we associate the diffeomorphism $G \in \mathrm{Diff}(S^1 \times D^q)$ defined by the formula: $G(x, y) := (x, g(y))$. Thus we obtain a map $K : \Theta_{q+1} \to \tilde{\pi}_0\mathrm{Diff}(S^1 \times D^q, \mathrm{rel}\ \partial)$ which is a monomorphism (see Proposition 6.2 of [10]). Since Θ_{q+2} is also a subgroup of $\tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial)$, we see that $\tilde{\pi}_0\mathrm{Diff}(S^1 \times D^q, \mathrm{rel}\ \partial) \cong \Theta_{q+1} \oplus \Theta_{q+2}$.

Gluck had shown in [1] (see §8 - §13) that $\tilde{\pi}_0\mathrm{Diff}(S^2 \times D^1, \mathrm{rel}\ \partial) \cong \mathbb{Z}_2$. As the generator of this group, one can take the homeomorphism T of $S^2 \times D^1$ defined by $T(x, t) := (f(t) \circ x, t)$, where $[f] \in \pi_1(SO(3))$ is as above. Since $\Theta_4 \cong \{0\}$, we can assume for the rest of our proof that $p \geq 3$. Since $\tilde{\pi}_0\mathrm{Diff}(M, \mathrm{rel}\ \partial) \hookrightarrow \tilde{\pi}_0 S\mathrm{Diff}(\mathcal{D}M)$ and using once again the generalized Dehn twist $(x, y) \to (\alpha(y) \circ x, y) \in S^p \times I$ with $[\alpha] =$ a smooth generator of $\pi_1(SO(p+1))$, it is easy to see that $B \circ \omega$ is an epimorphism, i.e. $\mathrm{Im}(B \circ \omega) = \mathbb{Z}_2$. Take $\varphi \in \mathrm{Diff}(S^p \times D^1, \mathrm{rel}\ \partial)$ such that $[\tilde{\varphi}] \in \mathrm{Ker}(B)$. According to Sato ([10], §6), we have the exact sequence

$$0 \longrightarrow \Theta_{p+1} \xrightarrow{K} \mathrm{Ker}(B) \xrightarrow{C} \Theta_{p+2} \longrightarrow 0$$

where map K was just defined in the paragraph above and C is the inverse map to the monomorphism ι_r which has been mentioned at the beginning (all the details regarding the homomorphism C can be found in §4 of [10]). If we have $[\tilde{\varphi}] \in \mathrm{Ker}(C)$, then we can assume that there exists $f \in \mathrm{Diff}(S^p \times S^1)$ such that $f \sim \tilde{\varphi}$ and $f(x, y) = (g(x), y)$ with $[g] \in \tilde{\pi}_0\mathrm{Diff}(D^q, \mathrm{rel}\ \partial) \cong \tilde{\pi}_0\mathrm{Diff}(S^q) \cong \Theta_{q+1}$. In this case we could extend $\tilde{\varphi}$ to a diffeomorphism of $S^p \times D^2$, that is $\tilde{\varphi} \sim Id$. Hence $\tilde{\pi}_0\mathrm{Diff}(S^p \times D^1, \mathrm{rel}\ \partial) \cong \Theta_{p+2} \oplus \mathbb{Z}_2$ as required. $\square$

Consider now a parallelizable $2p-$manifold F, which is obtained by gluing

μ handles of index $p \geq 2$ to the $2p-$disk and rounding the corners:

$$F = D^{2p} \cup \bigsqcup_{i=1}^{\mu} (D_i^p \times D^p)$$

Evidently $S^p \times D^p$ is an example of such a manifold. Given now a diffeomorphism φ of F which is the identity map on ∂F, one can consider the variation homomorphism $\mathrm{Var}(\varphi) : H_p(F, \partial F) \longrightarrow H_p(F)$ defined by the formula $\mathrm{Var}(\varphi)[z] := [f(z) - z]$ for any relative cycle $z \in H_p(F, \partial F)$ (cf. §1 of [11]). It is easy to show ([4], §2.2) that the elements of $\tilde{\pi}_0\mathrm{Diff}(F, \mathrm{rel}\ \partial)$ that induce zero variation homomorphism (i.e. $[\varphi]$ such that $\mathrm{Var}(\varphi)[z] = 0,\ \forall [z]$) form a subgroup of $\tilde{\pi}_0\mathrm{Diff}(F, \mathrm{rel}\ \partial)$. We will follow [4] and denote this subgroup by $\tilde{\pi}_0 V\mathrm{Diff}(F, \partial)$. Let us also denote by h the homomorphism $\tilde{\pi}_0\mathrm{Diff}(\mathcal{D}F) \longrightarrow \mathrm{Aut}\ H_p(\mathcal{D}F)$ induced by the natural action of the elements of $\tilde{\pi}_0\mathrm{Diff}(\mathcal{D}F)$ on the p-th homology of the double.

Claim 1.
Kernel of the homomorphism $h \circ \omega : \tilde{\pi}_0\mathrm{Diff}(F, \mathrm{rel}\ \partial) \longrightarrow \mathrm{Aut}\ H_p(\mathcal{D}F)$ *is equal to* $\tilde{\pi}_0 V\mathrm{Diff}(F, \partial)$.

Proof. It follows immediately from the proof of Theorem 1 of [4]. □

Let us denote by $S\pi_p(SO(p))$ the image of $\pi_p(SO(p))$ under the map $S_* : \pi_p(SO(p)) \longrightarrow \pi_p(SO(p+1))$ induced by the inclusion $SO(p) \hookrightarrow SO(p+1)$. Then $S\pi_6(SO(6))$ is trivial and for all other $p \geq 3$ the groups $S\pi_p(SO(p))$ are given in the table below (see [6], p. 644):

$p \pmod 8$	0	1	2	3	4	5	6	7
$S\pi_p(SO(p))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$

Lemma 3.

$$\tilde{\pi}_0\mathrm{Diff}(S^p \times D^p, \mathrm{rel}\ \partial) \cong \begin{cases} \mathbb{Z} & \text{if } \ p = 1 \\ \{1\} & \text{if } \ p = 2 \\ \Theta_{2p+1} \oplus S\pi_p(SO(p)) & \text{if } \ 4 \leq p \text{ is even} \\ \Theta_{2p+1} \oplus S\pi_p(SO(p)) \oplus \mathbb{Z} & \text{if } \ 3 \leq p \text{ is odd} \end{cases}$$

Proof. The case of $p = 1$ is well known and a proof can be found, for example, in §7 of [1]. When p is even, the image of $h : \tilde{\pi}_0\mathrm{Diff}(\mathcal{D}F) \longrightarrow \mathrm{Aut}\ H_p(S^p \times S^p)$ is isomorphic to $\mathbb{Z}_4$ and for any element $[\psi] \in \tilde{\pi}_0\mathrm{Diff}(\mathcal{D}F)$, $h([\psi])$ leaves no non-zero cycle of $H_p(S^p \times S^p)$ invariant (see Proposition 2.2 of [10]). Since the extended diffeomorphism $\tilde{\varphi}$ preserves $S^p \times D^p_+$ pointwise, it is clear that the group $\tilde{\pi}_0\mathrm{Diff}(F, \mathrm{rel}\ \partial)$ coincides with the kernel of $h \circ \omega$ and hence, by the claim above, $\tilde{\pi}_0\mathrm{Diff}(F, \mathrm{rel}\ \partial) \cong \tilde{\pi}_0 V\mathrm{Diff}(S^p \times D^p, \partial)$. The statement for p =even will now follow from the exact sequence (see [10], Theorem II)

$$0 \longrightarrow S\pi_p(SO(p)) \oplus \Theta_{2p+1} \longrightarrow \tilde{\pi}_0\mathrm{SDiff}(S^p \times S^p) \xrightarrow{B} S\pi_p(SO(p)) \longrightarrow 0,$$

a simple observation which has been already made that for every element $[\varphi] \in \tilde{\pi}_0 V\mathrm{Diff}(S^p \times D^p, \partial)$, the element $[\tilde{\varphi}]$ belongs to the kernel of B and Theorem 3 of [4] which says that

If $n = 2$ then $\tilde{\pi}_0 V\mathrm{Diff}(F, \partial) = 0$, and for all $n \geq 3$ the following sequence is exact

$$0 \longrightarrow \Theta_{2p+1} \xrightarrow{\iota_r} \tilde{\pi}_0 V\mathrm{Diff}(F, \partial) \longrightarrow \mathrm{Hom}(H_p(F, \partial F),\ S\pi_p(SO(p))) \longrightarrow 0.$$

Assume now that p is odd. Then $\mathrm{Aut}\ H_p(S^p \times S^p) \cong SL(2, \mathbb{Z})$ when p is 1, 3 or 7, and in the other cases $\mathrm{Aut}\ H_p(S^p \times S^p)$ is a proper subgroup of $SL(2, \mathbb{Z})$ which consists of the matrices $\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ such that both products d_1d_2 and d_3d_4 are even integers ([13], Lemma 5). Clearly, any matrix of this type is congruent modulo 2 either to $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We will denote this subgroup by $\Gamma_V(2)$. It is well known that $\Gamma_V(2)$ is not a normal subgroup of index 3 of $SL(2, \mathbb{Z})$ (see [9], §1.5). Moreover, using the fact that the corresponding projective group $\Gamma_V(2)/\mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}$ is generated by V and $T := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (cf. [13], §3) one can easily show that $\Gamma_V(2)$ admits the following presentation $\Gamma_V(2) \cong \langle V, T \mid V^4 = id,\ V^2T = TV^2 \rangle$. Assume that $p \geq 3$. It follows again from the definition of ω that the image of $h \circ \omega$ consists of those automorphisms that preserve the class of $H_p(S^p \times S^p)$ represented by an embedded sphere $S^p \times \{*\} \subset S^p \times D^p_+ \subset S^p \times S^p$. If we choose two spheres $S^p \times \{*\}$ and $\{*\} \times S^p$ as the basis of $H_p(S^p \times S^p)$, we see that $\mathrm{Im}(h \circ \omega)$ is generated either by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (when $p = 3$ or $p = 7$)

or by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (in the other cases) and hence $\mathrm{Im}(h \circ \omega) \cong \mathbb{Z}$. As for the corresponding element of $\tilde{\pi}_0 \mathrm{Diff}(M, \mathrm{rel}\ \partial)$, one can again take the generalized twist ϑ of $S^p \times D^p$ defined by the formula $\vartheta(x, y) := (\zeta(y) \circ x, y)$, where $(x, y) \in S^p \times D^p$ and $\zeta : (D^p, S^{p-1}) \longrightarrow (SO(p+1), Id)$ is a smooth map which generates image of the map $j_* : \pi_p(SO(p+1)) \to \pi_p(S^p)$ from the exact homotopy sequence of the fibration $SO(p) \hookrightarrow SO(p+1) \xrightarrow{j} S^p$. To finish the proof we need to show that for $p =$odd we also have $\tilde{\pi}_0 V\mathrm{Diff}(S^p \times D^p, \partial) \cong \Theta_{2p+1} \oplus S\pi_p(SO(p))$. If $p \geq 5$, one can use exactly the same argument which we gave above for $p =$even and if $p = 3$, it is shown in Example 1 of [4] that $\tilde{\pi}_0 V\mathrm{Diff}(S^3 \times D^3, \partial) \cong \Theta_7 \oplus \mathbb{Z}$. □

Let us now summarize what we have proved above and state the main result of this paper.

Theorem 1.

$$\tilde{\pi}_0 \mathrm{Diff}(S^p \times D^q, \mathrm{rel}\ \partial) \cong \begin{cases} \mathbb{Z} & \text{if } \ p = q = 1 \\ \{1\} & \text{if } \ 1 \leq p \leq 2, q = 2 \\ \Theta_{p+q+1} \oplus S\pi_p(SO(p)) & \text{if } \ 4 \leq p = q \text{ is even} \\ \Theta_{p+q+1} \oplus S\pi_p(SO(p)) \oplus \mathbb{Z} & \text{if } \ 3 \leq p = q \text{ is odd} \\ \Theta_{p+q+1} \oplus \Theta_{q+1} & \text{if } \ p = 1, q \geq 3 \\ \Theta_{p+q+1} \oplus \mathrm{FC}_q^{p+1} & \text{if } \ 1 < p < q \\ \Theta_{p+q+1} \oplus \pi_q(SO(p+1)) & \text{if } \ 1 \leq q < p \end{cases}$$

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